研究生入學能力考試試題(範例)

科目： 工程數學

考試日期：

1. If the 2nd order linear ordinary differential equation $y''+a(x)y'+b(x)y=0$ has two linearly independent solutions $y_1(x)$, and $y_2(x)$, is the linear combination $c_1y_1(x)+c_2y_2(x)$ also a solution of the equation? Why?
2. Is the set of functions $1, x^2, x^4$ linear independent or dependent for all $x\neq 0$? Why?
3. Find the general solution of the following ordinary differential equations
   (a) $2xydx-(2x^2-3y^2)dy=0$
   (b) $(y^2+xy^3)dx+(5y^2-xy+y^3\sin y)dy=0$
   (c) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^x+\sin x$
4. Find the solution of the following problem
   (a) $(\frac{1}{1+y^2}+\cos x-2xy)\frac{dy}{dx}=y(y+\sin x),\quad y(0)=1$
   (b) $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x,\quad y(0)=4$
5. A drug is infused into a patient's blood-stream at a constant rate of α grams per second. Simultaneously, the drug is removed at a rate proportional to the amount $M(t)$ of the drug present at time t. Determine a differential equation for $M(t)$.
6. A particle starts from the rest at the highest point of a vertical circle

with radius R, and slides under only the influence of gravity along a chord to another point on the circle. Show that the time taken is independent of the choice of the terminal point. What is the common time?

7. If a straight line $L_2: \ a_1x + a_2y = c,$ find a straight line L_1 which is perpendicular to L_2 and passes through the point $P(x_0, y_0)$.

8. Calculate the **area of the triangle** with vertices at P(3, 1, 6), Q(2, -3, 4) and R(7, -2, 4).

9. Find an equation for the plane passing through $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

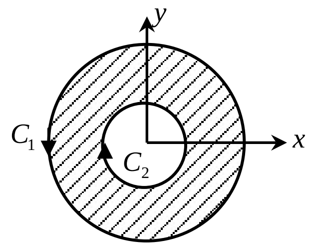
10. Find the **streamlines** of the velocity function

$$\mathbf{V}(x, y, z) = x^2\mathbf{i} + 2y\mathbf{j} - \mathbf{k}.$$

11. Given a force function $\mathbf{F} = y^2\mathbf{i} + (2xy + 2z)\mathbf{j} + 2y\mathbf{k}$, find the **potential** ϕ associated with $\mathbf{F}$.

12. Using **Stoke's Theorem**, find $\oint_\gamma \mathbf{v} \cdot d\mathbf{r}$, where $\mathbf{v}(x, y, z) = (z - 2y)\mathbf{i} + (3x - 4y)\mathbf{j} + (z + 3y)\mathbf{k}$, and γ is the unit circle in the plane $z = 2$.

13. Using **divergence theorem** , find $\oiint_B \mathbf{u} \cdot \mathbf{n} d\sigma$ if $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$ over the entire closed surface B bounded by the cylinder $x^2 + y^2 = a^2$, and the planes $z = 0$, $z = b$.



14. Using **plane Green's theorem**, Evaluate $\oint_C \mathbf{u} \cdot d\mathbf{r}$ if $\mathbf{u} = xy\mathbf{i} - x\mathbf{j}$, C: $C_1 \cup C_2$

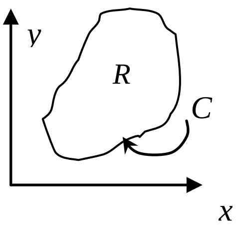
where $C_1 : x^2 + y^2 = 16, \quad C_2 : x^2 + y^2 = 4.$

15. (a) The temperature on a rectangular plate is given by $T(x, y) = 5 + 2x^2 + y^2$. Determine the **direction** an insect should take, starting at (4, 2), in order to cool off as rapidly as possible.

(b) Find the path the cold-seeking, insect will take starting at (4, 2), to the origin (0, 0).

16. Prove that the line integral $\int_C (Pdx + Qdy + Rdz)$ is independent of path in D if and only if it is zero on every close path in D.

17. Using plane Green's theorem, prove the area of R bounded by a simple closed curve C is given by



$$A = \frac{1}{2}\oint_C (xdy - ydx).$$

18. Consider a flow of fluid, from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

deduce that for potential flow (incompressible and irrotational)

$\nabla^2 \phi = 0$, where ϕ is called the velocity potential.

19. Are the set of vectors $\underset{\sim}{a}_1 = (1,\ 1,\ 1)$, $\underset{\sim}{a}_2 = (2,\ 3,\ 4)$, $\underset{\sim}{a}_3 = (3,\ 4,\ 5)$ linearly dependent or independent?

20. If $\vec{A}$ is a $m \times n$ matrix, and $\vec{B}$ is a $n \times q$ matrix, prove that

$$(\vec{A}\vec{B})^T = \vec{B}^T \vec{A}^T .$$

21. If $\vec{A}$ and $\vec{B}$ are both invertible $n \times n$ matrices. Show that

$$(\overleftrightarrow{A}\overleftrightarrow{B})^{-1} = \overleftrightarrow{B}^{-1}\overleftrightarrow{A}^{-1}.$$

22. (a) Is the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 23 & 54 & 6 \\ 0 & -15 & 21 \end{pmatrix}$ **invertible**?

(b) Using the formula $\overleftrightarrow{A}^{-1} = \dfrac{1}{|\overleftrightarrow{A}|}\text{adj}\,\overleftrightarrow{A}$, find $\overleftrightarrow{A}^{-1}$ of

$$\overleftrightarrow{A} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$$

23. If the matrix $\overleftrightarrow{A}$ is symmetric, prove that its **eigenvalues are all rea**l; and the **eigenvectors corresponding to distinct eigenvalues are orthogonal.** (10%)

24 Calculate the eigenvalues and the eigenvectors of the given matrix

$$\overleftrightarrow{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

25. Use the **method of eigenvector expansion**, solve $\overleftrightarrow{A}\underset{\sim}{x} - \underset{\sim}{x} = \underset{\sim}{c}$,

where $\overleftrightarrow{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (10%)

26. Consider a system of linear algebraic equations

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 4 \end{cases}.$$

(a) Find the rank of the coefficient matrix $\overleftrightarrow{A}$ and augmented matrix $\overleftrightarrow{G}$.

(b) Can you find the solution of the system? Why?

27. Is the set of functions $\left\{\cos\dfrac{n\pi x}{l},\ \ n=1,\ 2,\ 3,\ \cdots\right\}$ complete over the interval $[0,\ l]$? Why？

28. Given an equation $Lu+\lambda r(x)u=0,\ \ a<x<b$ -(1)

where L is a self-adjoint operator, $L=\dfrac{d}{dx}\left[p(x)\dfrac{d}{dx}\right]-q(x)$,

and λ is a real parameter, $q(x)\geq 0$, $r(x)>0$, are continuous on $[a,b]$, $p(x)>0$, is continuously differentiable on $[a,b]$. Then equation (1) is called a regular Sturm-Liouville equation.

(a) Identify the Bessel's equation and Legendre's equation are regular S-L equation

$$u''+\frac{1}{x}u'+(\alpha^2-\frac{n^2}{x^2})u=0,\quad \text{on}\quad 1<x<a\quad \text{(Bessel's eq.)}$$

$$(1-x^2)u''-2xu'+n(n+1)u=0,\quad \text{on}\quad -1<x<1\quad \text{(Legendre's eq.)}$$

(b) Equation (1) together with the boundary conditions $u(a)=u(b)=0$ constitute a regular S-L problem. Prove that

(i) Eigenfncs. corresponding to distinct eigenvalue are orthogonal w. r. t. the weight function. $r(x)$;

(ii) If $r(x)\neq 0$, in $a<x<b$, the eigenvalues are all real.

29. For the regular Sturm-Liouville problem

$$u''+\lambda u=0,\qquad 0<x<\pi$$

$$u(0)=0,\qquad u'(\pi)=0$$

(a) For nontrivial solution, what is the condition for λ?

(b) Find the eigenvalues and the eigenfunctions.

30. Use the method of eigenfunction expansion, find the solution of the

following inhomogeneous problem

$$u'' + 4u = 1, \qquad u(0) = u'(1) = 0.$$

31. Find power series solution for the following equation about the ordinary point $x = 0$.

$$(x+2)y'' + xy' - y = 0$$

32. Find a Fourier sine series for $f(x) = x^2 - 2$ in $0 < x < 2$ and use this series to obtain a series for π^3.

33. The Laplace Transform of a function $f(x)$ is defined as

$$\bar{f}(p) = \mathscr{L}\left[f(x); p\right] = \int_0^\infty f(x)e^{-px}dx.$$

(a) Prove (i) $\mathscr{L}\left[f''(x); p\right] = p^2\bar{f}(p) - pf(0) - f'(0);$

(ii) $\mathscr{L}\left[xf'(x); p\right] = -\frac{d}{dp}\mathscr{L}\left[f'(x); p\right] = -\frac{d}{dp}\left[p\bar{f}(p) - f(0)\right].$

(b) Use the results of (a) to solve the following initial value problem :

$$u'' + xu' + u = 0, \qquad x > 0$$
$$u(0) = 1, \quad u'(0) = 0.$$

where $$f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{for } t > 1 \end{cases}.$$

34. (a) Obtain $\mathscr{L}\left[e^{at}; p\right] = \frac{1}{p-a}$, then use this result to deduce

$$\mathscr{L}\left[\sin at; p\right] = \frac{a}{p^2 + a^2}, \qquad \mathscr{L}\left[\cos at; p\right] = \frac{p}{p^2 + a^2}.$$

(b) (10%) Use Laplace Transform method and the convolution theorem to solve $$f(t) + 2\int_0^t f(\tau)\cos(t-\tau)d\tau = \sin t.$$

35. Solve the initial value problem of 1st order linear partial differential equation

$$u_x - u_y + u = x, \quad \text{with I. C.} : u = x \ \text{ on } \ y = x.$$

How about the solution if the I. C. is : $u = x$ on $y = -x$?

37. Determine the regions in the *xy*-plane for which the equation

$$(xy+1)\frac{\partial^2 u}{\partial x^2} + (x+2y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, parabolic, or elliptic.

38. Solve the following initial-boundary value problem

Equation: $u_t = \kappa u_{xx}$, $0 < x < l,$ $t > 0$

Initial condition : $u(x,0) = u_0$ (const.);

Boundary conditions : $u(0,t) = 0,$ $u'(l,t) = 0.$

39. Solve the following initial-boundary value problem

Equation: $u_{tt} = c^2 u_{xx}$, $0 < x < l,$ $t > 0$

Initial conditions : $u(x,0) = f(x),$ $u_t(x,0) = 0;$

Boundary conditions : $u(0,t) = 0,$ $u(l,t) = 0.$

40. Solve the following boundary value problem of Laplace equation in a circular region :

Equation: $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$ $0 < r < r_0,$ $0 \le \theta \le 2\pi$

Boundary condition: $u(r_0,\theta) = f(\theta),$ $0 \le \theta \le 2\pi$